

Fixed Points of Automorphisms of Free Groups

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The set of fixed points of an automorphism of a finitely generated free group is a finitely generated group, settling a conjecture of G. P. Scott's. © 1987 Academic Press, Inc.

INTRODUCTION

A conjecture attributed to G. P. Scott [11] states that if ϕ is an automorphism of a finitely generated free group F , then $\text{Fix}(\phi) = \{x \in F \mid \phi(x) = x\}$ is finitely generated (see also [1, p. 195]). In this article we give an affirmative solution to Scott's conjecture. The solution is constructive in the sense that an algorithm is given to generate the fixed points of ϕ .

The history of the problem begins, to our knowledge, with a result of J. Dyer and G. P. Scott [1], who show that if ϕ is periodic, then in fact $\text{Fix}(\phi)$ is a free factor of F . Consequently, $\text{rank } \text{Fix}(\phi) \leq \text{rank } F$. Next there is a result of W. Jaco and P. Shalen [6], who show that $\text{Fix}(\phi)$ is finitely generated of rank at most that of F if ϕ is induced by a homeomorphism of the pair (M, x) , where M is a compact bounded orientable two dimensional manifold, $x \in M$. We took up the problem in [3], where we showed Scott's conjecture was true for ϕ induced by change of maximal tree (CMT, for short) in a finite connected graph X (defined in 1.8 below). Our result was constructive and actually gave an algorithm for producing the fixed points. In the CMT case, $\text{rank } \text{Fix}(\phi) \leq \text{rank } F$ is also true.

We strongly urge the reader to become acquainted with [3] before reading this paper, since the technical difficulties in the CMT case are minimal and all but two key ideas, to be explained below, are present there. This paper is written pretty much independently of [3], however.

Our idea is to decompose a general automorphism ϕ of F as a composition of Whitehead automorphisms [13]. This can be done by an algorithm due to J. H. C. Whitehead. We use a result of ours [3, 5.5] to represent a Whitehead automorphism as a CMT automorphism. Consequently we must analyze a composition of finitely many CMT

automorphisms. This is handled by a construction we call the “big fibre product,” 2.4. Sections 2 and 3 below are devoted to studying the properties of the big fibre product and its edge morphisms. These results are applied in 4.5 to give a graphical description of ϕ :

4.5. THEOREM. *Let Y be a finite graph with one vertex and let ϕ be an automorphism of $\pi_1(Y)$. Then there is a finite connected graph X , morphisms $f, f': X \rightarrow Y$, and a vertex v in X such that f and f' are special (Definition 3.4), their degenerate sets $D(f)$ and $D(f')$ each contain maximal trees of X , $\text{Ker } f_* = \text{Ker } f'_*$, where $f_*, f'_*: \pi_1(X, v) \rightarrow \pi_1(Y)$, and $f'_* \circ f_*^{-1} = \phi$ as automorphisms of $\pi_1(Y)$.*

In Section 5 we perform modifications on f, f' in 4.5 to produce a simpler graphical description of ϕ via morphisms $f, f': X \rightarrow Y$ satisfying properties

- G1: Df and Df' are maximal trees of X with no edges in common,
- G2: $f \mid (X - E(Df))$ and $f' \mid (X - E(Df'))$ are both immersions, and
- G3: f and f' are special with $\text{Ker } f_* = \text{Ker } f'_*$; $f'_* \circ f_*^{-1} = \phi$.

These properties G1–G3 and their formal consequence G4 (Theorem 5.10) are the only properties used in the remainder of the article. Property G4 is especially important; we call it “path surgery.” Along with the “big fibre product,” path surgery is the new idea beyond the CMT case [3] which enables us to solve Scott’s conjecture. In Section 6, we use path surgery to show (6.7) that fixed points of ϕ can be generated by the set \mathcal{M} of (f, f') reduced paths (Definition 6.2). Section 7 gives the algorithm for producing all elements of \mathcal{M} (7.6). The set \mathcal{M} is in general infinite (Example 6.8).

In Section 8 we discuss the algorithm 7.5 in more depth. A finite collection of paths in \mathcal{M} , the “standard paths” (8.1), is introduced. The main results here are 8.6 and 8.7 and are of a technical nature. They enable us to extract from \mathcal{M} a finite subset \mathcal{M}' (Definition 9.1) which is proved in 9.4 to generate all fixed points of ϕ . The main conclusion is drawn in 9.5, that $\text{Fix } \phi$ is finitely generated, and a bound on the rank is given in 9.7.

We deduce in 9.12, 9.13 that the fixed point set of any automorphism of a Fuchsian group, or more generally of a non-Euclidean crystallographic (NEC) group, is finitely generated.

The results of this article were announced in [4].

I wish to acknowledge my gratitude to John Stallings, who acquainted me with the problem and offered me encouragement as I came to understand it better.

Contents. 1. Graphs. 2. Fibre products. 3. Special morphisms. 4. Graphical representation of automorphisms, I. 5. Graphical representation of automorphisms, II. 6. Fixed points. 7. Construction of \mathcal{M} . 8. Further properties of \mathcal{M} . 9. Finiteness result. Appendix.

1. GRAPHS

We review the notion of graph introduced in [2]. We refer also to [10] and [9] at various times for proofs of the fundamental results.

1.1. A graph X is a nonempty set with involution $x \rightarrow \bar{x}$ (so $\bar{\bar{x}} = x$) and a retraction $\iota: X \rightarrow V(X)$, where $V(X) = \{x \in X \mid \bar{x} = x\}$, the fixed point set of the involution. Thus $\iota(\iota(x)) = \iota(x)$. We define $\tau(x) =: \iota(\bar{x})$ and $E(X) =: X - V(X)$. Intuitively $V(X)$ is to be thought of as the set of vertices, $E(X)$ the set of edges, $\iota(x)$ the initial vertex of x and $\tau(x)$ the terminal vertex. The edges $E(X)$ occur in pairs $x \neq \bar{x}$, $x \in E(X)$, an idea introduced by Serre [9].

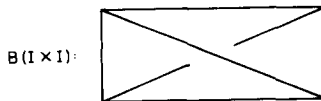
A morphism of graphs $f: X \rightarrow X'$ is a map f such that $f(\bar{x}) = \overline{f(x)}$ and $\iota(f(x)) = f(\iota(x))$. The category of graphs has a terminal object denoted $*$, fibre products, and push-outs. In addition, there is a geometrical realization functor $X \mapsto BX$ which assigns to the graph X a 1-dimensional CW complex and to a morphism a regular cellular map [2, 1.3]. This permits us to use the geometrical language of maximal trees, fundamental group, and homotopy equivalence.

For example, a subgraph X_1 of the connected graph X is a maximal tree if BX_1 is a maximal tree in BX . This notion of maximal tree can be described purely combinatorially [2, 1.6] without reference to BX .

The product in the category of graphs is somewhat peculiar. If I denotes the graph with BI the unit interval, we can take $I = \{z \in \mathbb{C} \mid z = 0, 1, 1/2 + i, 1/2 - i\}$, where the involution is complex conjugation.

$$V(I) = \{0, 1\}, \quad \iota(1/2 + i) = 0, \quad \iota(1/2 - i) = 1.$$

Then $I \times I$ has 4 vertices and 12 edges, the latter paired under the involution.



1.2. A *path* p [2, 2.1] in the graph X is an ordered n tuple ($n \geq 1$), $p = (x_1, x_2, \dots, x_n)$, $x_i \in X$, such that $\tau(x_j) = \iota(x_{j+1})$, $1 \leq j \leq n-1$. Let $l(p) = n$, the *length* of the path p . The vertices $\iota(x_1)$ and $\tau(x_n)$ are called the

initial and terminal points of p , written $\iota(p)$ and $\tau(p)$; respectively. There is the *trivial path* v at a vertex v . If two paths p and p' are such that $\iota(p') = \tau(p)$, then their composite $p \cdot p'$ is defined by concatenation. The operation is associative when defined.

An *elementary reduction* $p \searrow p'$ of paths is either

1.2.1. $p = (x_1, \dots, x_n)$ is replaced by $p' = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$ if $x_j \in V(X)$ and $n > 1$, or

1.2.2. $p = (x_1, x_2, \dots, x_n)$ is replaced by $p' = (x_1, \dots, x_{j-1}, i(x_j), x_{j+2}, \dots, x_n)$ if $x_{j+1} = \bar{x}_j$.

The path p is called *reduced* if it admits no elementary reductions. Two paths, p and p' , are called *equivalent*, written $p \sim p'$, if there is a finite sequence of paths $p = p_1, p_2, \dots, p_m = p'$ such that either $p_j \searrow p_{j+1}$ or $p_{j+1} \searrow p_j$ for $1 \leq j \leq m-1$.

1.3. PROPOSITION [10]. (a) *Each path is equivalent to a unique reduced path.*

(b) *The operation of composition of paths is compatible with equivalence. That is, $p \sim p'$, $q \sim q'$ implies $p \cdot q \sim p' \cdot q'$ if the compositions are defined.*

1.4. We recall the combinatorial description of the fundamental group [9]. If X is a graph and $v \in V(X)$, $\pi_1(X, v)$ is the set of equivalence classes of paths p such that $\iota(p) = \tau(p) = v$ (such a path is called a *circuit* at v). It is a group under composition of paths, and the inverse of the class represented by $p = (x_1, \dots, x_n)$ is represented by $\bar{p} =: (\bar{x}_n, \bar{x}_{n-1}, \dots, \bar{x}_1)$. $\pi_1(X, v)$ is a free group whose rank is the rank of $H_1(X_0)$, where X_0 is the connected component of X containing v .

To exhibit a free basis for $\pi_1(X, v)$ proceed as follows. Choose an orientation \mathcal{O} [10] on X . Thus $\mathcal{O} \subset E(X)$ and $\#(\mathcal{O} \cap \{x, \bar{x}\}) = 1$ for each $x \in E(X)$. Let T be a maximal tree of X_0 . Denote by $[v, w]_T$ the (unique) reduced path in T from vertex v to vertex w ; $v, w \in V(X_0)$. Then a free basis for $\pi_1(X, v)$ is given by the equivalence classes of the circuits

$$[v, \iota(e)]_T \cdot e \cdot [\tau e, v]_T$$

for $e \in \mathcal{O} \cap (X_0 - T)$.

If $f: X \rightarrow X'$ is a morphism of graphs then f induces a map of paths $p = (x_1, \dots, x_n) \rightarrow fp = (f(x_1), f(x_2), \dots, f(x_n))$. This map preserves equivalence classes of paths and respects the composition operation, so induces a homomorphism $f_{*,v}: \pi_1(X, v) \rightarrow \pi_1(X', f(v))$, where $v \in V(X)$. If the vertex v is understood, we shall often abbreviate $f_{*,v}$ to f_* .

1.5. If X is a graph and $v \in V(X)$, let $\text{Star}_X(v) = \{x \in X \mid \iota(x) = v\}$. A morphism $f: X \rightarrow X'$ of graphs induces a map $f_v: \text{Star}_X(v) \rightarrow \text{Star}_{X'}(f(v))$. We say f is an *immersion* [10, 3.1] if f_v is injective for each $v \in V(X)$. If each f_v is bijective, f is a *covering*. The main properties of immersions are summarized in

1.6. PROPOSITION [10, Sect. 5]. *Let $f: X \rightarrow X'$ be an immersion.*

- (a) *If p is a reduced path in X , then fp is a reduced path in X' .*
- (b) *If p and q are paths in X with $\iota(p) = \iota(q)$ and $fp = fq$, then $p = q$.*
- (c) *If $v \in V(X)$, the induced map $f_*: \pi_1(X, v) \rightarrow \pi_1(X', f(v))$ is injective.*

1.7. If $f: X \rightarrow X'$ is a morphism of graphs, we define the *degenerate set* $Df = \{x \in X \mid f(x) \in V(X')\}$. Df is always a subgraph of X . If $Df = V(X)$, we say f is *non-degenerate*. The morphisms considered by Stallings [10] and Serre [9] are what we call non-degenerate morphisms.

1.8. EXAMPLE. Let X be a connected graph and let T be a maximal tree in X . Let the diagram

$$\begin{array}{ccc} T & \subseteq & X \\ \downarrow & & \downarrow \rho \\ * & \longrightarrow & X/T = Y \end{array}$$

be a push out diagram. Then $D\rho = T$ and ρ is a homotopy equivalence of X with the one vertex graph Y , so $\rho_*: \pi_1(X, v) \xrightarrow{\cong} \pi_1(Y)$ for any $v \in V(X)$. If T' is another maximal tree in X and if the diagram

$$\begin{array}{ccc} T' & \subseteq & X \\ \downarrow & & \downarrow \rho' \\ * & \longrightarrow & Y \end{array}$$

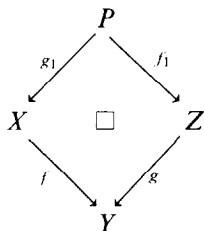
is also a push out diagram, then the automorphism ϕ_v of $\pi_1(Y)$ given by the composition

$$\pi_1(Y) \xrightarrow{\rho_*^{-1}} \pi_1(X, v) \xrightarrow{\rho'_*} \pi_1(Y)$$

is called a CMT (for “change of maximal tree”) automorphism of $\pi_1(Y)$. Such automorphisms and their fixed points were studied in [3].

2. FIBRE PRODUCTS

2.1. PROPOSITION. Suppose $f: X \rightarrow Y$ and $g: Z \rightarrow Y$ are morphisms of connected graphs where Y has one vertex. Suppose Df and Dg contain maximal trees of X and Z , respectively. In the fibre product diagram



it follows that P is connected and $D(f \circ g_1)$ contains a maximal tree of P . If in addition f is surjective, so is f_1 .

Proof. Since $\#V(Y) = 1$, it follows that $V(P) = V(X) \times V(Z)$. Let (x, z) and $(x_1, z_1) \in V(P)$. Since x and x_1 can be connected in Df , (x, z) and (x_1, z) can be connected in $Df \times \{z\}$. Similarly (x_1, z) and (x_1, z_1) can be connected in $\{x_1\} \times Dg$. Since $Df \times \{z\}$ and $\{x_1\} \times Dg$ are contained in $D(f \circ g_1)$, the latter is connected and contains all vertices of P . Thus P is connected and $D(f \circ g_1)$ contains a maximal tree of P .

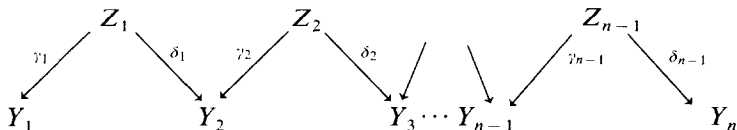
Suppose f is surjective and let $z \in Z$. If $f(x) = g(z)$ with $x \in X$, then $(x, z) \in P$ and $f_1((x, z)) = z$, so f_1 is surjective. This completes the proof of 2.1.

2.2. Remark. The argument showed that for any $z \in V(Z)$, $f_1^{-1}(z) = Df \times \{z\}$ is connected.

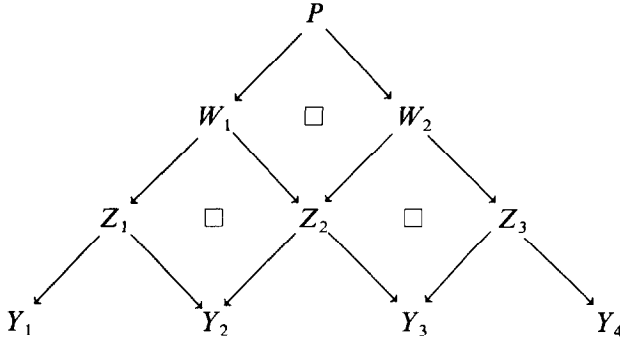
2.3. PROPOSITION. Suppose $P \xrightarrow{f} Z \xrightarrow{g} Y$ are morphisms of connected graphs where Y has one vertex. Suppose that Dg contains a maximal tree T of Z and f is surjective and that for all $z \in V(Z)$, $f^{-1}(z)$ is connected. Then $D(gf)$ contains a maximal tree of P .

Proof. Each $f^{-1}(z) \subset D(gf)$, for $z \in V(Z)$. For each $t \in T$ choose $e_t \in P$ with $f(e_t) = t$. Then $C = \bigcup_{z \in V(Z)} f^{-1}(z) \cup_{t \in T} \{e_t\}$ is a connected subgraph of P containing $V(P)$, and $C \subseteq D(gf)$.

2.4. Consider now the following diagram of morphisms:



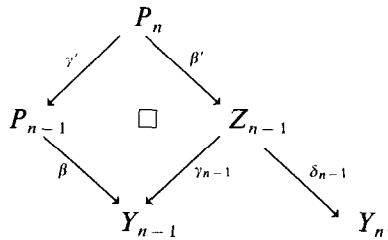
Here $\gamma_i: Z_i \rightarrow Y_i$ and $\delta_i: Z_i \rightarrow Y_{i+1}$, $1 \leq i \leq n-1$. We can fill in by taking successive fibre products to get a pyramidal diagram, illustrated below for $n=4$:



The top of the pyramid we call P and the composite morphism $P \rightarrow Y_i$ we call α_i . Observe that α_i is independent of the route chosen from P to Y_i . We call this construction, for want of better terminology, the “big fibre product.”

2.4.1. PROPOSITION. *Suppose that all Y_i and Z_i are connected and all Y_i have one vertex. Assume each γ_i, δ_i is surjective and each $D(\gamma_i), D(\delta_i)$ contains a maximal tree of Z_i . Then P is connected, $D(\alpha_1)$ and $D(\alpha_n)$ contain maximal trees of P , and $P \rightarrow Y_1$ and $P \rightarrow Y_n$ are surjective.*

Proof. We proceed by induction on n . Call P_j the top of the pyramid constructed from the data γ_i, δ_i for $i \leq j-1$. Then $P_2 = Z_1$ and the conclusion holds for $j=2$ by the hypothesis. Assume the result of $j=n-1$ and consider a portion of the big fibre product diagram:



By the induction hypothesis, P_{n-1} is connected, $D(\beta)$ contains a maximal tree of P_{n-1} , and β is surjective. From 2.1 and 2.2, the fibres of β' over vertices of Z_{n-1} are connected, β' is surjective, and P_n is connected. From 2.3, $D(\delta_{n-1} \circ \beta')$ contains a maximal tree of P_n . Also, $\delta_{n-1} \circ \beta'$ is surjective. But $\alpha_n = \delta_{n-1} \circ \beta'$, so the conclusions are valid for α_n . To get them for α_1 , just reason by symmetry.

2.5. *Remark.* We shall need to impose even stronger conditions on the γ_i, δ_i in the next section to guarantee that the maps $\alpha_i: P \rightarrow Y_i$ are surjective on π_1 .

3. SPECIAL MORPHISMS

In this section, we consider composites of edge collapses and edge folds.

3.1. **DEFINITION.** An *edge collapse* is a push out diagram

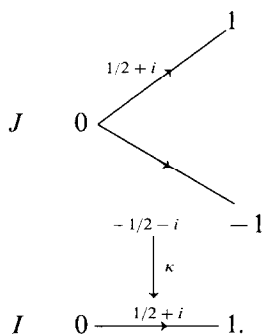
$$\begin{array}{ccc} I & \xrightarrow{j} & X \\ \downarrow & & \downarrow f \\ * & \longrightarrow & Y \end{array}$$

where j is a morphism of the combinatorial unit interval $I(1.1)$ into X . If $e = j(1/2 + i)$, then the effect of f is to identify $e, \bar{e}, \iota(e)$, and $\tau(e)$ to one vertex.

Define a graph J as follows. Let $J = \{z \in \mathbb{C} \mid z = 0, \pm 1, \pm 1/2 \pm i\}$ and let the involution on J be complex conjugation. Define $\iota: J \rightarrow V(J)$ by

$$\iota(1/2 + i) = \iota(-1/2 - i) = 0, \quad \iota(1/2 - i) = 1, \quad \iota(-1/2 + i) = -1.$$

The geometrical realization of J is two intervals with a vertex in common:



Define a map $\kappa: J \rightarrow I$ by $\kappa(z) = z$ if $\operatorname{Re} z \geq 0$, $\kappa(z) = -z$ if $\operatorname{Re} z \leq 0$. The effect of $B\kappa$ is to fold the two intervals to one interval.

We define an *edge fold* to be a push out diagram

$$\begin{array}{ccc} J & \xrightarrow{j_1} & X \\ \kappa \downarrow & & \downarrow f \\ I & \xrightarrow{j_2} & Y \end{array}$$

where j_1 and j_2 are morphisms. If j_1 is non-degenerate and $j_1(1/2 + i) \neq j_1(-1/2 + i)$, this diagram is admissible in the sense of [10, 3.2]. If on the other hand $j_1(1/2 + i) = j_1(-1/2 + i)$, then f is in fact an edge collapse.

3.2. PROPOSITION. *If $X \rightarrow^f Y$ is an edge collapse or an edge fold, then for any $v \in V(X)$, $f_*: \pi_1(X, v) \rightarrow \pi_1(Y, f(v))$ is surjective.*

Proof. The argument given in [10, 4.3] applies to this situation without change.

The significance of edge collapses and edge folds is made evident in the following:

3.3. PROPOSITION. *Any morphism $f: X \rightarrow Y$ of finite graphs factors as $f = f_3 \circ f_2 \circ f_1$, $X \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} Y$, where f_1 is a composition of edge collapses, f_2 is a composition of admissible edge folds, and f_3 is an immersion.*

Proof. Each connected component Z of Df is mapped by f to a vertex of Y . We can thus factor f as gf_1 ,

$$X \xrightarrow{f_1} X_1 \xrightarrow{g} Y$$

where f_1 occurs in a push out diagram

$$\begin{array}{ccc} Df & \subseteq & X \\ \downarrow & & \downarrow f_1 \\ U & \longrightarrow & X_1, \end{array}$$

where U is a disjoint union of copies of $*$, one for each component of Df , and $Df \rightarrow U$ maps each component of Df to its corresponding component of U .

The map $g: X_1 \rightarrow Y$ is non-degenerate, so g factors as $f_3 \circ f_2$, where f_2 is a composition of admissible edge folds and f_3 is an immersion [10, 3.3]. In addition, it is easy to see that f_1 is a composition of edge collapses. This completes the proof.

3.4. DEFINITION. A morphism $f: X \rightarrow Y$ of finite graphs is called *special* if it is a composition of edge collapses and edge folds. Special morphisms

are surjective on π_1 , computed at any vertex in X , by 3.2. Observe also that any special morphism is surjective, since edge folds and edge collapses are surjective.

3.5. THEOREM. *Let $f: X \rightarrow Y$ be a morphism of finite connected graphs. Then there is a factorization*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \rho \searrow & & \nearrow j \\ & Z & \end{array}$$

where ρ is special and j is an immersion. In addition the factorization is unique in the sense that if

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \rho' \searrow & & \nearrow j' \\ & Z' & \end{array}$$

is another factorization with ρ' special and j' an immersion, then there is an isomorphism $\sigma: Z \rightarrow Z'$ such that $\sigma\rho = \rho'$, $j'\sigma = j$,

$$\begin{array}{ccccc} & & Z & & \\ & \nearrow \rho & \downarrow \sigma & \searrow j & \\ X & & & & Y \\ & \searrow \rho' & \downarrow & \nearrow j' & \\ & & Z' & & \end{array}$$

Proof. The existence of the factorization $f = j \circ \rho$ follows from 3.3.

For uniqueness, suppose we have two factorizations of f , $f = j\rho = j'\rho'$, where ρ, ρ' are special and j, j' are immersions. Construct P , the fibre product of j and j' , and complete to a commutative diagram, using the universal property of fibre products:

$$\begin{array}{ccccc} & & X & & \\ & \nearrow \rho & \downarrow \tau & \searrow \rho' & \\ Z & & P & & Z' \\ & \xleftarrow{\sigma} & & \xrightarrow{\sigma'} & \\ & & \square & & \\ & \searrow j & & \nearrow j' & \\ & & Y & & \end{array}$$

Observe that σ and σ' are immersions, since the pull-back of an immersion is also an immersion. Since X is connected and ρ, ρ' are surjective, it follows that Z, Z' are connected. If $v \in V(X)$, ρ_* and ρ'_* , the induced maps on π_1 , are surjective (3.2). It follows that σ_* and σ'_* , the induced maps on $\pi_1(P, \tau(v))$ are surjective. But σ, σ' are immersions, so σ_*, σ'_* are injective (1.6c), hence isomorphisms. In addition σ, σ' are surjective maps (since ρ, ρ' are surjective).

Admit for the moment the validity of the following:

3.6. LEMMA. *If $f: X \rightarrow Y$ is an immersion of finite connected graphs which is surjective and such that $f_*: \pi_1(X, v) \rightarrow \pi_1(Y, fv)$ is an isomorphism for some $v \in V(X)$, then f is an isomorphism.*

Returning to the proof of 3.5, let P_0 be the connected component of P containing $\tau(v)$. We see that $\sigma|P_0$ and $\sigma'|P_0$ are isomorphisms of graphs. It follows that $(\sigma'|P_0) \circ (\sigma|P_0)^{-1}$ is the desired isomorphism $Z \rightarrow Z'$ and the proof of 3.5 is complete, assuming Lemma 3.6.

Proof of Lemma 3.6. The version of M. Hall's theorem due to J. Stallings (see the Appendix) implies there is a commutative diagram

$$\begin{array}{ccc} & X \subseteq X' & \\ & \swarrow \quad \searrow & \\ f & & f' \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

where X is a subgraph of X' and f' is a finite connected covering. Since f_* is an isomorphism, f'_* is also an isomorphism. But f' is a connected covering, so f' is an isomorphism. In particular this implies f is injective. Since f is also surjective, f is an isomorphism of graphs. The proof of 3.6 is complete.

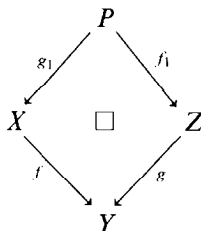
3.7. COROLLARY. *If $f: X \rightarrow Y$ is a surjective morphism of finite connected graphs and $v \in V(X)$, then $f_*: \pi_1(X, v) \rightarrow \pi_1(Y, fv)$ is surjective iff f is special.*

Proof. We have $f = j \circ \rho$, where ρ is special and j is an immersion. Since f is surjective, it follows that j is surjective. If f_* is surjective, then j_* is an isomorphism. By 3.6, j is an isomorphism and it follows that f is special.

The result just proved gives the desired homotopy characterization of special morphisms.

3.8. PROPOSITION. *Suppose $f: X \rightarrow Y$ and $g: Z \rightarrow Y$ are morphisms of finite connected graphs where Y has one vertex. Assume f and g are special*

and Df and Dg contain maximal trees of X and Z respectively. In the fibre product diagram,



it follows that P is connected, $f \circ g_1$ is special, and $D(f \circ g_1)$ contains a maximal tree of P . If in addition g is a homotopy equivalence, then f_1 is special.

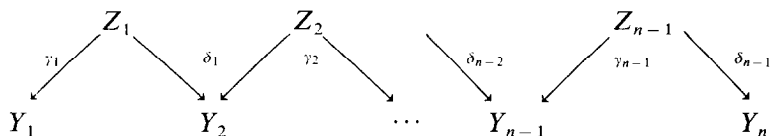
Proof. That P is connected and $D(f \circ g_1)$ contains a maximal tree T of P follows from 2.1. Since f is special, it is surjective, whence f_1 is surjective. Thus $g \circ f_1 = f \circ g_1$ is surjective. Let y be an edge of Y and let $x \in E(P)$ with $f \circ g_1(x) = y$. Let $v \in V(P)$. Let ξ be the class of the circuit

$$[v, \iota x]_T \cdot x \cdot [\tau x, v]_T$$

in $\pi_1(P, v)$ and let η denote the class of y in $\pi_1(Y)$ (since Y has only one vertex). Then $T \subset D(f \circ g_1)$ so $(f \circ g_1)_*(\xi) = \eta$. Thus $(f \circ g_1)_*$ is surjective on π_1 . It follows from 3.7 that $f \circ g_1$ is special.¹

If in addition g is a homotopy equivalence, then $(f_1)_*$ is surjective. Since f_1 is surjective, it follows from 3.7 that f_1 is special. This completes the proof of 3.8.

3.9. We return again to the situation of 2.4, with a diagram



and fill in to form the pyramidal diagram of the big fibre product P and the edge morphisms $\alpha_1: P \rightarrow Y_1$, $\alpha_n: P \rightarrow Y_n$.

¹ John Stallings informed me that this result follows from a *Lemma*: If $f: X \rightarrow Y$ is a surjective morphism and Df is connected, then f is special.

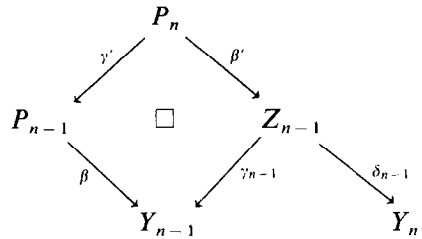
3.9.1. PROPOSITION. *Suppose all Y_i and Z_i are connected and Y_i are one vertex graphs. Assume each $D\gamma_i$, $D\delta_i$ is a maximal tree of Z_i and that γ_i, δ_i factor as*

$$\begin{aligned} Z_1 &\longrightarrow Z_1/D\gamma_1 \xrightarrow{\cong} Y_1, \\ Z_i &\longrightarrow Z_i/D\delta_i \xrightarrow{\cong} Y_{i+1} \end{aligned}$$

where the first map collapses a maximal tree and the second map is an isomorphism. Then the edge morphisms $\alpha_1: P \rightarrow Y_1$, $\alpha_n: P \rightarrow Y_n$ in the big fibre product are special, P is connected, and $D(\alpha_1)$ and $D(\alpha_n)$ contain maximal trees of P .

Proof. That P is connected and the $D(\alpha_1)$, $D(\alpha_n)$ contain maximal trees of P follow from 2.4.1. To see α_1 and α_n are special, we proceed by induction on n , the induction starting because γ_1 and δ_1 are special.

In the inductive step, assume P_{n-1} is the top of the pyramid constructed from γ_i, δ_i , $i \leq n-2$ and consider the diagram



Here β is an edge map of the big fibre product for step $n-1$, so β is special. Since γ_{n-1} is special and a homotopy equivalence, it follows from 3.8 that β' is special. Then $\alpha_n = \delta_{n-1} \circ \beta'$ is special. That α_1 is special follows from symmetry. This completes the proof of 3.9.1.

4. GRAPHICAL REPRESENTATION OF AUTOMORPHISMS, I

In this section we show how to represent an automorphism of a finitely generated free group by the edge maps of a big fibre product (2.4).

4.1. Let ϕ be an automorphism of $\pi_1(Y)$, where Y is a finite graph with only one vertex. If \mathcal{O} is an orientation of Y (Section 1) then \mathcal{O} gives a free basis for $\pi_1(Y)$. With respect to this basis, we can define Whitehead automorphisms of $\pi_1(Y)$ [7, p. 31; 3, 5.4]. A fundamental result of J. H. C. Whitehead's [13] asserts $\phi = \tau_{n-1} \circ \tau_{n-2} \circ \cdots \circ \tau_1$, where τ_i are Whitehead automorphisms of $\pi_1(Y)$.

In [3, 5.4] we showed that each Whitehead automorphism τ_i was a CMT automorphism of Y . This means there is a connected finite (two-vertex) graph Z_i , morphisms $Y \xleftarrow{\gamma_i} Z_i \xrightarrow{\delta_i} Y$, where $D\gamma_i, D\delta_i$ are maximal trees of Z_i , γ_i and δ_i factor as

$$Z_i \longrightarrow Z/D\gamma_i \xrightarrow{\cong} Y,$$

$$Z_i \longrightarrow Z/D\delta_i \xrightarrow{\cong} Y$$

respectively; and there is a vertex v_i in Z_i such that $\delta_{i*} \circ \gamma_{i*}^{-1}: \pi_1(Y) \rightarrow \pi_1(Y)$ is precisely τ_i , when the fundamental group of Z_i is computed at v_i .

Hence we have a diagram

$$\begin{array}{ccccccc} & Z_1 & & Z_2 & & Z_{n-2} & & Z_{n-1} \\ & \swarrow \gamma_1 & \searrow \delta_1 & \swarrow \gamma_2 & \searrow \delta_{n-2} & \swarrow \gamma_{n-1} & \searrow \delta_{n-1} \\ Y = Y_1 & & Y_2 & \cdots & & Y_{n-1} & & Y_n = Y \end{array}$$

where all $Y_i = Y$, and we can form the big fibre product P and its edge maps $\alpha_1: P \rightarrow Y$, $\alpha_n: P \rightarrow Y$. These are special by 3.9.1. In addition, the vertices v_i in Z_i determine a vertex v in P , so we have surjective homomorphisms $\alpha_{1*}, \alpha_{n*}: \pi_1(P, v) \rightarrow \pi_1(Y)$.

We shall see how ϕ can be recovered from these data.

4.2. LEMMA. $\text{Ker } \alpha_{1*} = \text{Ker } \alpha_{n*}$.

Proof. Denote by α_i the (unique) map $P \rightarrow Y_i$ in the pyramid. Let β_i denote the (unique) map $P \rightarrow Z_i$. Then $\alpha_i = \gamma_i \circ \beta_i$, $\alpha_{i+1} = \delta_i \circ \beta_i$. Since γ_{i*} and δ_{i*} are isomorphisms, $\text{Ker } \alpha_{i*} = \text{Ker } \beta_{i*} = \text{Ker } \alpha_{(i+1)*}$. Thus $\text{Ker } \alpha_{1*} = \text{Ker } \alpha_{n*}$.

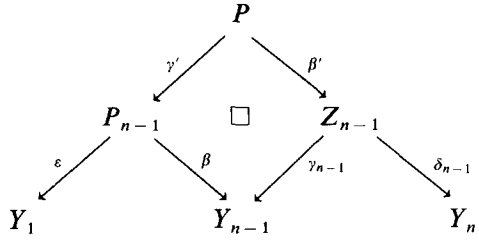
4.3. Suppose in general that f and g are two homomorphisms of a group G onto a group H such that $\text{Ker } f = \text{Ker } g$. Then we can define an automorphism ψ of H by $g \circ f^{-1}$. Equivalently f and g determine canonical isomorphisms $\bar{f}, \bar{g}: G/N \xrightarrow{\cong} H$, where $N = \text{Ker } f = \text{Ker } g$, and $\psi = \bar{g} \circ \bar{f}^{-1}$.

We can apply this construction to α_{1*}, α_{n*} in 4.2 to get an automorphism $\psi = \alpha_{n*} \circ \alpha_{1*}^{-1}$ of $\pi_1(Y)$.

4.4. PROPOSITION. In this setting, $\phi = \psi$ as automorphisms of $\pi_1(Y)$.

Proof. We proceed by induction on n . Since $\tau_1 = \delta_{1*} \circ \gamma_1^{-1}$, the induction starts.

Assuming the result for $n-1$, let P_{n-1} be the top of the pyramid constructed from γ_i, δ_i , $i \leq n-2$. We have a diagram



Here ε, β are the edge maps of the pyramid for step $n-1$, so $\beta_* \varepsilon_*^{-1} = \tau_{n-2} \circ \cdots \circ \tau_1$, by induction. Also $\alpha_1 = \varepsilon \circ \gamma'$ and $\alpha_n = \delta_{n-1} \beta'$ and $\psi = \alpha_n \circ \alpha_1^{-1} = \alpha_n \circ (\varepsilon \circ \gamma')_*^{-1} = \alpha_n \circ \gamma'^{-1} \varepsilon_*^{-1} = \alpha_n \circ \gamma'^{-1} \beta_*^{-1} \beta_* \varepsilon_*^{-1} = \delta_{(n-1)*} \beta'_* \circ (\gamma'^{-1} \circ \beta_*^{-1}) \beta_* \circ \varepsilon_*^{-1} = \delta_{(n-1)*} \beta'_* (\beta'^{-1} \circ \gamma_{(n-1)*}^{-1}) \circ \beta_* \circ \varepsilon_*^{-1} = \delta_{(n-1)*} \circ \gamma_{(n-1)*}^{-1} \circ \beta_* \circ \varepsilon_*^{-1} = \tau_{n-1} \circ (\tau_{n-2} \circ \cdots \circ \tau_1) = \phi$.

We can organize our results so far in

4.5. THEOREM. *Let Y be a finite graph with one vertex and let ϕ be an automorphism of $\pi_1(Y)$. There is a finite connected graph X , morphisms $f, f': X \rightarrow Y$, and a vertex $v \in V(X)$ such that*

- 4.5.1. f and f' are special,
- 4.5.2. $D(f)$ and $D(f')$ contain maximal trees of X ,
- 4.5.3. $\text{Ker } f_* = \text{Ker } f'_*$ when $f_*, f'_*: \pi_1(X, v) \rightarrow \pi_1(Y)$, and
- 4.5.4. $f'_* \circ f_*^{-1} = \phi$ as automorphisms of $\pi_1(Y)$.

Proof. We factor ϕ as a composition $\tau_{n-1} \circ \cdots \circ \tau_2 \circ \tau_1$ of Whitehead automorphisms and represent each τ_i as a CMT automorphism of $\pi_1(Y)$, then form the big fibre product P and edge maps $\alpha_1, \alpha_n: P \rightarrow Y, v \in V(P)$ as in 4.1. We already observed α_1 and α_n are special in 4.1. $D(\alpha_1)$ and $D(\alpha_n)$ contain maximal trees of P and P is connected by 3.9.1. $\text{Ker } \alpha_{1*} = \text{Ker } \alpha_{n*}$ by 4.2. Finally $\alpha_{n*} \circ \alpha_{1*}^{-1} = \phi$ by 4.4.

Thus $X = P$, $f = \alpha_1$, and $f' = \alpha_n$ satisfy the conclusions of 4.5.

5. GRAPHICAL REPRESENTATION OF AUTOMORPHISMS, II

In this section, we modify X, f, f' in 4.5 so that f and f' are immersions off their degenerate sets. We preserve the notation of 4.5 here.

5.1. PROPOSITION. *If Y is a finite graph with one vertex and ϕ is an automorphism of $\pi_1(Y)$, then X, f, f' in 4.5 may be chosen so that 4.5.1–4.5.4 are satisfied and in addition $E(Df) \cap E(Df') = \emptyset$.*

Proof. Let U be a disjoint union of copies of $*$, one for each component of $D(f) \cap D(f')$. Map $D(f) \cap D(f')$ to U by collapsing each connected

component to the corresponding point of U . Consider the push out diagram

$$\begin{array}{ccc} D(f) \cap D(f') & \xrightarrow{\subseteq} & X \\ \downarrow & & \downarrow \rho \\ U & \longrightarrow & X_1. \end{array}$$

Then f and f' factor through ρ to give $f_1, f'_1: X_1 \rightarrow Y$, respectively. Let $v_1 = \rho(v)$. We claim X_1, f_1, f'_1, v_1 satisfy the corresponding conditions 4.5.1–4.5.4. ρ_* is surjective on π_1 by repeated application of 3.2, and ρ itself is surjective from properties of push outs. Thus X_1 is connected and f_1 and f'_1 are special by 3.7. Since $D(f_1), D(f'_1)$ are images of $D(f), D(f')$ under ρ , they contain maximal trees of X_1 . If $N = \text{Ker } \rho_*$ and $K = \text{Ker } f_* = \text{Ker } f'_*$, then $N \subset K$ and $\text{Ker } f_{1*} = \text{Ker } f'_{1*} \cong K/N$. Lastly, $\phi = f'_* \circ f_*^{-1} = (f'_1 \circ \rho)_* \circ (f_1 \circ \rho)_*^{-1} = f'_{1*} \circ \rho_* \circ \rho_*^{-1} f_{1*}^{-1} = f'_{1*} \circ f_{1*}^{-1}$, so the properties 4.5.1–4.5.4 are satisfied for X_1, f_1, f'_1, v_1 .

5.2. By virtue of 5.1, we may assume that $E(D(f)) \cap E(D(f')) = \emptyset$ in addition to 4.5.1–4.5.4 in Theorem 4.5.

With this additional assumption, assume that $e, e' \in E(D(f'))$ are distinct edges folded by f . That is, $ie = ie'$ and $fe = fe' \in E(Y)$, (since $E(D(f)) \cap E(D(f')) = \emptyset$). Observe that the pair (e, e') is *admissible* [10, 3.2]. For if $e' = \bar{e}$, then $f(e') = f(\bar{e}) = \bar{f(e)}$ and $f(e') = f(e)$, so $f(e) = \bar{f(e)} \in V(Y)$; thus $e \in D(f)$, a contradiction. We can thus perform an admissible fold on X , identifying e and e' to form a quotient graph, X_1 , [10, 3.3]. Since $e, e' \in D(f')$, both f and f' factor through the quotient graph and the factorizations f_1, f'_1 are special morphisms $X_1 \rightarrow Y$. We wish to verify that f_1, f'_1 satisfy 4.5.1–4.5.4. The only non-trivial verification is 4.5.3, which we carry out.

The quotient map $X \rightarrow X_1$ is a homotopy equivalence unless $\tau e = \tau e'$ [10, 4.4]. In case $\tau e = \tau e'$, the kernel N of the quotient map $\rho: \pi_1(X, v) \rightarrow \pi_1(X_1, v)$ is normally generated by the equivalence class of the circuit $[v, ie]_T \cdot e \cdot \bar{e}' \cdot [ie, v]_T$, where T is a maximal tree of X in Df . Thus $N \subset \text{Ker } f_* = \text{Ker } f'_*$. From this it follows that $\text{Ker } f_{1*} = \text{Ker } f'_{1*}$.

5.3. After a finite number folds, each of which decreases the number of edges of X , we can achieve the situation that $f \mid Df': Df' \rightarrow Y$ and $f' \mid Df \rightarrow Y$ are both immersions, and 4.5.1–4.5.4 are satisfied, in addition, in 4.5. We observe an important consequence of this.

5.4. PROPOSITION. *Suppose that, in the situation of 4.5, $f \mid Df'$ and $f' \mid Df$ are both immersions. Then Df and Df' are maximal trees of X .*

Proof. By 4.5.2, Df contains a maximal tree T of X . If $e \in E(Df) - E(T)$, let ξ denote the equivalence class in $\pi_1(X, v)$ of the circuit

$$[v, ie]_T \cdot e \cdot [\tau e, v]_T.$$

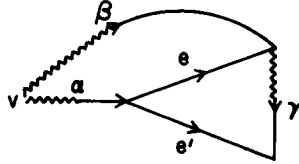
Since the circuit is in Df , $f_*\xi = 1$. But f' immerses Df in Y , so $\xi = 1$ in $\pi_1(X, v)$ by 1.6c. But this is absurd, since ξ is a free generator of $\pi_1(X, v)$ by 1.4. Thus $Df = T$. Similarly Df' is a maximal tree of X .

We need a stronger condition than the hypothesis of 5.4, which cannot be achieved by passage to a quotient in general.

5.5. PROPOSITION. Suppose X, f, f', v, Y are as in 4.5 and in addition $f|D(f')$ and $f'|D(f)$ are immersions. Let e, e' be distinct edges in $E(X) - E(D(f))$ be such that $ie = ie'$, $fe = fe'$, and $e \notin Df'$. Let $X_1 = X - \{e, \bar{e}\}$ and let f_1, f'_1 be the restrictions of f, f' to X_1 , respectively. Then X_1, f_1, f'_1, v, Y satisfy the conclusions of 4.5, namely 4.5.1–4.5.4.

Remark. If the distinct edges e, e' are folded by f , then one of them must not lie in $D(f')$, since $f|D(f')$ is assumed to be an immersion. Thus there's no loss in generality in assuming $e \notin D(f')$. The pair (e, e') is admissible, since if $\bar{e} = e'$, then $e \in D(f)$, contrary to hypothesis.

Proof of 5.5. Since Df is a maximal tree in X by 5.4, let $\alpha = [v, ie]_{Df}$, $\beta = [v, \tau e]_{Df}$, and $\gamma = [\tau e, \tau e']_{Df}$.



Since $fe = fe'$, if we let ξ be the equivalence class of the circuit

$$\alpha \cdot e \cdot \gamma \cdot \bar{e}' \cdot \bar{\alpha},$$

then $\xi \in \text{Ker } f_* = \text{Ker } f'_*$. We shall show that f_{1*} and f'_{1*} are surjective. Now the free generator of $\pi_1(X, v)$ corresponding to $e \notin Df$ is $\alpha \cdot e \cdot \beta$. But

$$f_*([\alpha \cdot e \cdot \beta]) = f_*([\alpha \cdot e \cdot \gamma \cdot \bar{e}' \cdot \bar{\alpha}] \cdot [\alpha \cdot e' \cdot \bar{\gamma} \cdot \bar{\beta}]) = f_*([\alpha \cdot e' \cdot \bar{\gamma} \cdot \bar{\beta}])$$

where “[]” denotes the equivalence class of a path. But $[\alpha \cdot e' \cdot \bar{\gamma} \cdot \bar{\beta}] \in \pi_1(X_1, v)$, since $Df \subset X_1$. Thus f_{1*} is surjective. The same computation shows that f'_{1*} is also surjective.

Since f_{1*} and f'_{1*} are surjective and Y has only one vertex, it follows that f_1 and f'_1 are surjective. For if f_1 , say, were not surjective, then the image

of f_{1*} would be contained in a proper free factor of $\pi_1(Y)$. Hence f_1 and f'_1 are special, by 3.7, and 4.5.1 is verified.

Since $D(f) = D(f_1)$ and $D(f') = D(f'_1)$, it is clear that 4.5.2 is satisfied for f_1, f'_1 .

To see that 4.5.3 is satisfied, let $G = \pi_1(X, v)$, $G_1 = \pi_1(X_1, v)$, and $K = \text{Ker } f_* = \text{Ker } f'_*$. Observe that G_1 is a subgroup (in fact, a free factor) of G . Then $\text{Ker } f_{1*} = K \cap G_1 = \text{Ker } f'_{1*}$, verifying 4.5.3.

The last property, 4.5.4, follows immediately from

5.5.1. LEMMA. *Let $f, f': G \rightarrow H$ be homomorphisms of groups with $\text{Ker } f = \text{Ker } f'$. Let $G_1 \leq G$ be such that $f|_{G_1}$ and $f'|_{G_1}$ are both surjective. The $f' \circ f^{-1} = f'_1 \circ f_1^{-1}$ as automorphisms of H .*

This completes the proof of 5.5.

It is clear that one may iterate the argument of 5.5, removing if necessary a finite number of edges of X , to achieve at the end the situation that $f|_{X-ED(f)}$ is an immersion and $f'|_{X-E(D(f'))}$ is an immersion. We state this result as

5.6. THEOREM. *Let Y be a finite graph with one vertex and let ϕ be an automorphism of $\pi_1(Y)$. Then there is a finite connected graph X , morphisms $f, f': X \rightarrow Y$ and $v \in V(X)$ such that*

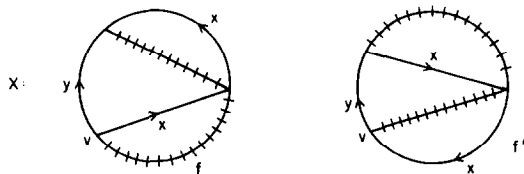
G1: Df and Df' are maximal trees of X and $E(Df) \cap E(Df') = \emptyset$,

G2: $f|_{(X-E(Df))}$ and $f'|_{(X-E(Df'))}$ are both immersions,

G3: f and f' are special, $\text{Ker } f_* = \text{Ker } f'_*$, and $f'_* \circ f_*^{-1} = \phi$.

Remark. We have attempted to axiomatize as many of the arguments that follow as possible. In fact, only G1–G3 and their consequence G4, to be introduced in 5.10, are used in describing fixed points of ϕ .

5.7. EXAMPLE. Let $\phi: F(x, y) \rightarrow F(x, y)$ be given by $x \mapsto x, y \mapsto yx^2$. We draw the geometrical realization of X in 5.6 realizing ϕ :

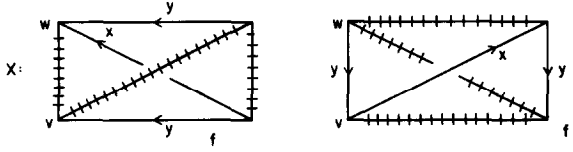


(Hatching indicates the maximal trees Df, Df' ; the labels indicate how f, f' map X to the bouquet



of two circles.)

5.8. EXAMPLE. Let $\phi: F(x, y) \rightarrow F(x, y)$ given by $x \rightarrow xy^2$, $y \rightarrow xy$. The geometrical realization of X realizing ϕ in 5.6 is drawn below:



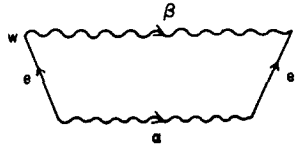
If one chooses the vertex w instead, one realizes the automorphism $x \rightarrow yxy$, $y \rightarrow yx$.

5.9. EXAMPLE. Let ϕ be the automorphism of $F(x, y, z)$ given by $x \rightarrow xy$, $y \rightarrow yz$, $z \rightarrow x^{-1}$. The pictures realizing ϕ in 5.6 are the following:



5.10. THEOREM. Suppose $f, f': X \rightarrow Y$ are morphisms satisfying G1–G3 of 5.6. Then property G4 below follows:

G4: (“Path Surgery”) Let $e, e' \in E(X) - E(Df)$ be such that $fe = fe'$. Then $l(\bar{e} \cdot \alpha \cdot e') > l(\beta)$, where $\alpha = [ie, ie']_{Df}$ and $\beta = [\tau e, \tau e']_{Df}$. A similar statement is valid with f' replacing f throughout.



Proof. Let $w = \tau e$. It is convenient to use w as base point for the fundamental group. Observe that since X is connected and $\text{Ker } f_{*v} = \text{Ker } f'_{*v}$ by G3, it follows that $\text{Ker } f_{*w} = \text{Ker } f'_{*w}$.

Observe that $\bar{e} \cdot \alpha \cdot e' \cdot \beta$ is a circuit at w such that $f_{*w}([\bar{e} \cdot \alpha \cdot e' \cdot \beta]) = 1$. Hence $f'_{*w}([\bar{e} \cdot \alpha \cdot e' \cdot \beta]) = 1$. Thus $f'(\bar{e}\alpha e') \simeq f'(\beta)$. However, $f' \mid Df$ is an immersion (by G1 and G2) and β is a reduced path in Df , so $f'(\beta)$ is the path of minimal length in its equivalence class. Hence

$$l(\beta) = l(f'(\beta)) \leq l(f'(\bar{e}\alpha e')) = l(\bar{e} \cdot \alpha \cdot e').$$

We claim the inequality here is strict. If not, $l(f'(\bar{e} \cdot \alpha \cdot e')) = l(f'(\beta))$, and $f'(\bar{e} \cdot \alpha \cdot e')$ is equivalent to $f'(\beta)$. It follows that $f'(\bar{e}\alpha e')$ is reduced and hence equal to $f'(\beta)$ (1.3).

Now $\bar{e}\alpha e'$ and β are two paths with the same initial point w . If $\beta = e_1 e_2 \cdots e_n$ ($e_i \in E(Df)$), we deduce that $f'(e_1) = f'(\bar{e})$. Now $e_1 \in Df$ but $\bar{e} \notin Df$ (by hypothesis). Since e_1 is mapped non-degenerately by f' , $\bar{e} \notin Df'$.

Thus $\bar{e}, e_1 \notin Df'$, $i\bar{e} = ie_1$, and $f'(\bar{e}) = f'(e_1)$. But $f' \mid (X - E(Df'))$ is an immersion by G2. Thus $\bar{e} = e_1$, so $e \in Df$, contrary to hypothesis.

Hence $l(\beta) < l(\bar{e} \cdot \alpha \cdot e')$ and the proof of 5.10 is complete.

6. FIXED POINTS

We define the notions that enable us to analyze fixed points of automorphisms. The notation is that of 5.6, and properties G1–G4 are valid for $f, f': X \rightarrow Y$.

6.1. DEFINITION. Let $\text{Fix}(\phi) = \{\eta \in \pi_1(Y) \mid \phi(\eta) = \eta\}$. Let $\mathcal{E} = \{\xi \in \pi_1(X, v) \mid f_*(\xi) = f'_*(\xi)\}$.

6.1.1. LEMMA. $f_*(\mathcal{E}) = \text{Fix}(\phi)$.

Proof. $f_*(\xi) \in \text{Fix}(\phi)$ iff $\phi f_*(\xi) = f_*(\xi)$ iff $(f'_* \circ f_*^{-1}) \circ f_*(\xi) = f_*(\xi)$ iff $f'_*(\xi) = f_*(\xi)$ iff $\xi \in \mathcal{E}$. \mathcal{E} maps onto $\text{Fix}(\phi)$ since f_* is surjective.

6.2. DEFINITION. A reduced path (x_1, x_2, \dots, x_n) in X is called *f-reduced* if, after deleting all $x_i \in Df$, then applying f to the remaining edges, the resulting path in Y is reduced. The path p in X is called *(f, f')-reduced* if it is both *f-reduced* and *f'-reduced*. A path p in X is called *invariant* if $f_*(p) = f'_*(p)$, when $f_*(p)$ denotes the homotopy class of $f(p)$. An *(f, f')-reduced invariant path* p in X is called *minimal* if no non-trivial initial segment of p is invariant. We denote by \mathcal{M} the set of minimal *(f, f')-reduced invariant paths*.

6.3. PROPOSITION. If $\eta \in \pi_1(Y)$, then there are only finitely many *f-reduced paths* p in X with $f_*p = \eta$.

Proof. Let η be represented by the (unique) reduced path (y_1, y_2, \dots, y_n) in Y . Let $x_i \in E(X)$ be such that $f(x_i) = y_i$ (these exist by G3). Let $w, w' \in V(X)$. Then the path $p = [w, ix_1]_{Df} \cdot x_1 \cdot [\tau x_1, ix_2]_{Df} \cdot x_2 \cdot \cdots \cdot x_n \cdot [\tau x_n, w']_{Df}$ is *f-reduced* and $f_*p = \eta$. Conversely, if p' is an *f-reduced path* with $f_*p' = \eta$, $ip' = w$, $\tau p' = w'$, then, since p' is reduced and Df is a maximal tree of X , p' can be written in the form $p' = [w, ix'_1]_{Df} \cdot x'_1 \cdot [\tau x'_1, ix'_2]_{Df} \cdot x'_2 \cdot \cdots \cdot x'_m \cdot [\tau x'_m, w']_{Df}$ with $x'_i \notin Df$. The condition that p' be *f-reduced* requires $n = m$ and $f(x'_i) = y_i$, $1 \leq i \leq n$.

Since there are only a finite number of choices of x_i , $f(x_i) = y_i$, and of $w, w' \in V(X)$, there are only finitely many *f-reduced paths* p with $f_*p = \eta$.

6.4. THEOREM. *For each reduced path p in X there exists an (f, f') -reduced path p_1 such that*

$$6.4.1. \quad \imath p = \imath p_1, \tau p = \tau p_1 \text{ and}$$

$$6.4.2. \quad f_* p = f_* p_1, f'_* p = f'_* p_1.$$

Proof. Suppose that p is not f -reduced. Then p must be a path product of reduced paths $p = p_0 \cdot p_1 \cdot p_2$, with “no cancellation” (so, e.g., if x is the last edge of p_0 and x' is the first edge of p_1 , then $\bar{x} \neq x'$), where p_0 and p_2 are possibly empty, and p_1 has the form

$$p_1 = e \cdot [\tau e, \imath e']_{Df} \cdot e',$$

with $e, e' \notin Df$ and $\bar{f}e = f(e')$. By G4 of 5.10, $l([\tau e, \imath e']_{Df}) < l(p_1)$. Thus if $p' = p_0 \cdot [\tau e, \imath e']_{Df} \cdot p_2$, then p' is a path with $\imath p' = \imath p$, $\tau p' = \tau p$, and $f_* p' = f_* p$. But this implies that $f_*(p' \cdot \bar{p}) = 1$, where $p' \cdot \bar{p} \in \pi_1(X, \imath p)$. Since X is connected, G3 (which was stated at vertex v) implies $f'_*(p' \cdot \bar{p}) = 1$, so $f'_*(p') = f'_*(p)$. In addition $l(p') < l(p)$.

Since p' need not be reduced, we may apply elementary reductions to reduce it, to get a reduced path p'' having the same end points as p and the same images under f_* and f'_* as p , and in addition with $l(p'') < l(p)$.

If p'' is either not f -reduced or not f' -reduced, we can continue with an analogous argument using “path surgeries,” G4. Each surgery decreases the length, so the process converges in a finite number of steps with an (f, f') -reduced path satisfying the conclusions of the theorem.

6.5. COROLLARY. *If $\eta \in f_*(\mathcal{E})$, there exists an (f, f') -reduced circuit w based at v such that w is invariant and $f_* w = \eta$.*

Proof. Let $w' \in \mathcal{E}$ be such that $f_* w' = \eta$. By 6.4 there is an (f, f') -reduced circuit w based at v with $f_* w = f_* w' = \eta$, $f'_* w = f'_* w' = \eta$, so w is invariant.

Remark. If w is an (f, f') reduced invariant path in X , then clearly $w = w_1 \cdot w_2 \cdots w_r$, where $w_i \in \mathcal{M}$, and we can even assume there is “no cancellation” in this product. It is convenient to rephrase 6.5 to make use of this fact.

6.6. Let Z be the graph obtained from X by identifying all vertices of X to one vertex. Formally, we have a push out diagram

$$\begin{array}{ccc} V(X) & \xrightarrow{\quad \quad} & X \\ \downarrow & & \downarrow \rho \\ * & \longrightarrow & Z \end{array}$$

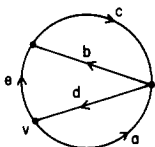
where $X \rightarrow^\rho Z$ is an immersion (since ρ is clearly bijective on edges). By 1.6c, the induced map $\rho_*: \pi_1(X, v) \rightarrow \pi_1(Z)$ is injective, so we can identify $\pi_1(X, v)$ with its image under ρ_* . Similarly, \mathcal{M} consists of reduced paths of X , so by 1.6a, \mathcal{M} can be identified with a subset of $\pi_1(Z)$.

Furthermore, the maps $f, f': X \rightarrow Y$ factor through Z , since Y has only one vertex. Denote the factorizations $Z \rightarrow Y$ by the same letters f, f' . With these conventions we have

6.7. COROLLARY. $f_*(\mathcal{E}) = f_*(\langle \mathcal{M} \rangle \cap \pi_1(X, v))$, where $\langle \mathcal{M} \rangle$ denotes the subgroup generated by \mathcal{M} .

Proof. That the left-hand side is contained in the right-hand side follows from 6.5 and the Remark following it. Conversely, suppose $\xi \in \langle \mathcal{M} \rangle \cap \pi_1(X, v)$ (all taking place in $\pi_1(Z)$). This implies $f_*(\xi) = f'_*(\xi)$, so $\xi \in \mathcal{E}$. This $f_*(\xi) \in f_*\mathcal{E}$, and the proof is complete.

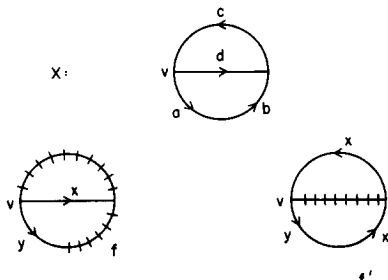
6.8. EXAMPLE. Recall example 5.7, $\phi: x \rightarrow x, y \rightarrow yx^2$. Label X there as follows (only oriented edges are drawn):



Then $\mathcal{M} = \{e, \bar{e}, da, \bar{a}\bar{d}, bc, \bar{c}\bar{b}, a \cdot (bc)^n \cdot d, \bar{d}(\bar{c}\bar{b})^n \bar{a}, c(da)^n b, \bar{b}(\bar{a}\bar{d})^n \bar{c}; n \geq 0\}$. From this, one computes

$$\text{Fix}(\phi) = f_*\mathcal{E} = f_*(\langle \mathcal{M} \rangle \cap \pi_1(X, v)) = \langle x, yxy^{-1} \rangle.$$

6.9. Remark. I have been asked why a description as complicated as 5.7, 6.8 is necessary to study as innocuous an automorphism as $\phi: x \rightarrow x, y \rightarrow yx^2$. Perhaps I can convince the reader that this example is not nearly so innocuous by considering the graph X , whose realization is drawn below, along with two maps f, f' to the bouquet of two circles.

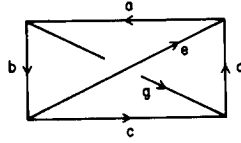


Observe that $f'_{*v} \circ f_{*v}^{-1} = \phi$. Here $\mathcal{M} = \{cd, \bar{d}\bar{c}, dc, \bar{c}\bar{d}, a, \bar{a}\}$, so $f_*(\langle \mathcal{M} \rangle \cap \pi_1(X, v)) = \langle x \rangle$. In this case there is no (f, f') -reduced invariant circuit representing $xyx^{-1} \in \text{Fix}(\phi)$. For example, the circuit $abcd\bar{b}\bar{a} = p$ is such that $f_*p = f'_*p = yxy^{-1}$, and p is f -reduced. But p is not f' -reduced and p cannot be modified by surgery to give an (f, f') -reduced circuit.

What goes wrong in this example is that Df' is not a maximal tree of X and G1 is not satisfied (although G2 and G3 are valid and Df is a maximal tree of X). Nor is G4, the formal consequence of G1–G3, satisfied.

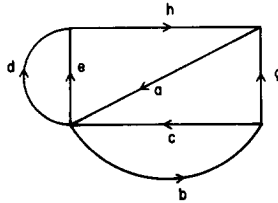
6.10. EXAMPLE. We state what \mathcal{M} is in Example 5.8, although the techniques to justify this are postponed until 7.8.

Label X in 5.8:



Then $\mathcal{M} = \{ab, \bar{b}\bar{a}, cd, \bar{d}\bar{c}\}$. Hence for $\phi: x \rightarrow xy^2, y \rightarrow xy$ one has $\text{Fix}(\phi) = f_*(\langle \mathcal{M} \rangle \cap \pi_1(X, v)) = \{1\}$.

6.11. EXAMPLE. Return to Example 5.9 and label X there:



Then $\mathcal{M} = \{ab, \bar{b}\bar{a}, cd\bar{e}, e\bar{d}\bar{c}\}$ although this will be justified only later. Hence $\text{Fix}(\phi) = \{1\}$.

7. CONSTRUCTION OF \mathcal{M}

In this section we discuss a procedure for constructing the elements of \mathcal{M} . We assume throughout that $f, f': X \rightarrow Y$ satisfy conditions G1–G3 of 5.6.

7.1. LEMMA. Let $p \in \mathcal{M}$ and let e be the initial edge of p . Then either $e \in E(Df)$, or $e \in E(Df')$, or $fe = f'e \in E(Y)$.

Proof. Assume $e \notin Df \cup Df'$ and $fe = x$. Then x is the initial edge in the reduced word $f_*p = f'_*p$ since p is (f, f') -reduced. Then $fe = f'e$.

We shall follow the convention that if $e, e' \in E(X)$, then $e \cdot [\tau e, \iota e']_{Df} \cdot e' = e \cdot e'$ if $\tau e = \iota e'$. In other words, we omit the trivial path $\tau e = \iota e'$ from $e \cdot [\tau e, \tau e']_{Df} \cdot e'$ if $e \cdot e'$ is defined.

In analyzing $p \in \mathcal{M}$, we may restrict ourselves thus to p with initial edge e in Df . The case with initial edge in Df' is handled by interchanging roles of f and f' , and the case $e \in E(X) \cap \mathcal{M}$ requires no further elaboration.

7.2. LEMMA ("Necessary condition"). *If $p \in \mathcal{M}$ has initial edge $e \in Df$ and $x_1 \in E(X)$ is the first edge (reading from left to right) in p satisfying $x_1 \notin Df$, then $f'e = fx_1$ and $[\tau e, \iota x_1]_{Df}$ does not have initial edge \bar{e} . Thus $e \cdot [\tau e, \iota x_1]_{Df} \cdot x_1 = [\iota e, \iota x_1]_{Df} \cdot x_1$ is (f, f') -reduced.*

Proof. We can write $p = p_1 \cdot p_2$, where $p_1 = e_1 \cdot e_2 \cdot \dots \cdot e_n \cdot x_1$ and $e_1 = e$, $e_i \in Df$. Since p is (f, f') -reduced, so is p_1 . But f_*p begins with $f_*p_1 = fx_1$ and f'_*p begins with $f'e_1 = f'e$. Thus $f'(e) = f(x_1)$.

We now establish a partial converse.

7.3. LEMMA. *If $e \in E(Df)$ and $x_1 \in E(X)$ is such that $f(x_1) = f'(e)$ and $[\tau e, \iota x_1]_{Df}$ does not begin with \bar{e} , then*

$$p_1 =: e \cdot [\tau e, \iota x_1]_{Df} \cdot x_1$$

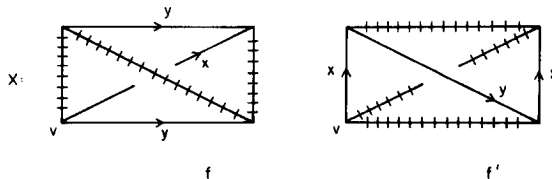
is (f, f') -reduced.

Proof. It remains only to check that p_1 is f' reduced. This is clear if $x_1 \in E(Df')$ since $f' \mid Df$ is an immersion and $e \cdot [\tau e, \iota x_1]_{Df}$ is reduced by hypothesis. If $x_1 \in E(X) - E(Df')$, it will follow (since $f' \mid (X - ED(f'))$ is an immersion) once we know that p_1 is reduced. But $x_1 \notin Df$ so no "cancellation" can take place between $[\tau e, \iota x_1]_{Df}$ and x_1 , and no "cancellation" takes place between e and $[\tau e, \iota x_1]_{Df}$ by hypothesis. This completes the proof.

We shall regard p_1 as a first stage in an algorithm to produce a path $p \in \mathcal{M}$ with initial edge e .

7.3.1. EXAMPLE. The results 7.2, 7.3 give a necessary condition for there to exist $p \in \mathcal{M}$ starting with a given edge e . We refer the reader to [3] for examples in the CMT case, 1.8 where this necessary condition cuts down the size of \mathcal{M} . Here is a non-CMT example.

Let φ be given by $x \mapsto \bar{x}\bar{y}\bar{x}$, $y \mapsto \bar{y}\bar{x}$. Then φ is realized geometrically by X, f, f' below:



It is easily checked that no element of \mathcal{M} starts in Df , using 7.2. Similarly, one checks that no element of \mathcal{M} starts in Df' . Thus $\mathcal{M} = \emptyset$, and thus $\text{Fix } \varphi = \{1\}$, by 6.7 and 6.1.1.

7.4. Remark. In the CMT case [3], the necessary condition 7.1 was the only obstruction to the existence of a *potentially infinite* (f, f') -reduced invariant minimal word. In effect, this followed since reduced paths are automatically (f, f') reduced in the CMT case. This is no longer true in the general setting.

7.5. If the hypotheses of 7.3 are satisfied, then $f_*p_1 = fx_1 =: y_1$ and $f'_*p_1 = y_1y_2 \cdots y_{s_1}$ (reduced). If $s_1 = 1$, then $p_1 \in \mathcal{M}$ and we're done. If $s_1 \geq 2$, pick $x_2 \in E(X)$ such that $f(x_2) = y_2$ and let

$$p_2 = p_1 \cdot [\tau p_1, ix_2]_{Df} \cdot x_2.$$

This path is f -reduced and $f_*p_2 = y_1y_2$. If p_2 is also f' -reduced, then $f'_*p_2 = y_1 \cdots y_{s_1} \cdot y_{s_1+1} \cdots y_{s_2}$ (reduced), where $s_2 \geq s_1$. If $s_2 = 2$, then $p_2 \in \mathcal{M}$. If $s_2 \geq 3$, we pick x_3 such that $f(x_3) = y_3$ and form $p_3 = p_2 \cdot [\tau p_2, ix_3]_{Df} \cdot x_3$, again an f -reduced path which may fail to be f' -reduced. Continuing in this way we construct a (potentially infinite) sequence of "approximate invariant" paths p_1, p_2, \dots such that p_n is an initial segment of p_{n+1} . Having constructed p_n , (f, f') -reduced, we may only be obstructed from constructing p_{n+1} if (1) for all choices of x_{n+1} with $f'x_{n+1} = y_{n+1}$, the path $p_{n+1} = p_n \cdot [\tau p_n, ix_{n+1}]_{Df} \cdot x_{n+1}$ is not f' reduced, or (2) if p_n is itself in \mathcal{M} (i.e., we have exhausted all the y_i 's, $f_*p_n = f'_*p_n$).

That this procedure produces all elements of \mathcal{M} is the content of the next result.

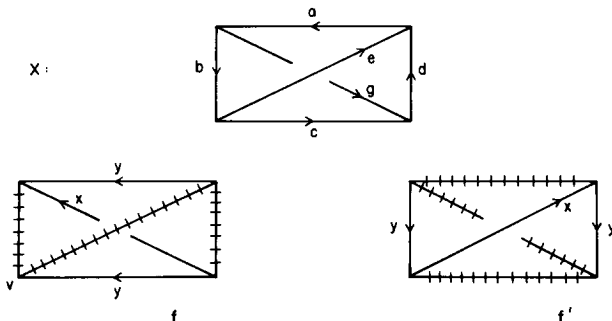
7.6. PROPOSITION. Suppose $p \in \mathcal{M}$ has initial edge $e \in Df$. Then there is a finite sequence of "approximate invariants" p_1, p_2, \dots, p_n as in 7.5 with $p_n = p$.

Proof. Write $f_*p = f'_*(p) = y_1 \cdots y_n$, a reduced path in Y . Then there are precisely n edges in p , x_i , $1 \leq i \leq n$, which are not in Df , and with $fx_i = y_i$ (since p is f -reduced). Thus p can be written uniquely as $p = [ip, ix_1]_{Df} \cdot x_1 \cdot [\tau x_1, ix_2]_{Df} \cdot x_2 \cdots x_n \cdot [\tau x_n, \tau p]_{Df}$. Define p_j to be the initial segment of p up through x_j , so $f_*p_j = y_1y_2 \cdots y_j$ and $f'_*p_j = y_1y_2 \cdots y_{s_j}$. Observe that $j \leq s_j \leq s_{j+1}$ and $j = s_j$ if and only if p_j is invariant. Since $p \in \mathcal{M}$, p is a minimal invariant and hence $p = p_n$, so $\tau(x_n) = \tau p$ as well. This completes the proof of 7.6.

7.7. COROLLARY ("Antisymmetry Principle"). If $p \in \mathcal{M}$ has its initial edge in Df , then p has its last edge in Df' .

Proof. In the notation of the proof of 7.6, $p = p_n$ has its last edge $x_n \notin Df$. If we consider \bar{p} , it is clear $\bar{p} \in \mathcal{M}$, so by 7.1, \bar{x}_n is either in Df or in Df' or else $f\bar{x}_n = f'\bar{x}_n \in E(Y)$. But the last alternative would contradict minimality of p , for then a proper nontrivial initial segment of p would be invariant. Thus $\bar{x}_n \in Df'$ and hence $x_n \in Df'$. This completes the proof.

7.8. EXAMPLE. Return to the example 5.8, 6.10, q.v.



One checks that all the edges of Df satisfy the necessary condition 7.2. Among the edges of Df' , only a and c satisfy the necessary condition and $ab, cd \in \mathcal{M}$. We claim that these two paths ab, cd are the only elements of \mathcal{M} starting in Df' . For example, the only choices for x_1 in 7.5 with $f'(x_1) = f(a) = y$ are b and d . However, $[\tau a, id]_{Df'} = \bar{a}$, so $a \cdot [\tau a, id]_{Df'} \cdot \bar{d}$ is not reduced. Hence \bar{d} is eliminated and only $ab \in \mathcal{M}$ starts with a . The reasoning for cd is similar. We deduce that $\bar{b}\bar{a}, \bar{d}\bar{c} \in \mathcal{M}$ from this.

If any element $p \in \mathcal{M}$ begins in Df , it must end in Df' by 7.7, hence must end in \bar{a} or \bar{c} . Thus \bar{p} begins in a or c , so $\bar{p} = ab$ or cd . Thus $p = \bar{b}\bar{a}$ or $\bar{d}\bar{c}$. In particular, no path in \mathcal{M} starts with b, d, e or \bar{e} .

Thus $\mathcal{M} = \{ab, \bar{b}\bar{a}, cd, \bar{d}\bar{c}\}$ as claimed in 6.10.

The argument for \mathcal{M} in 6.11, 5.9 is similar and is omitted.

Thus 7.2 and 7.7 give useful tools for limiting the size of \mathcal{M} , whereas 7.6 provides an algorithm for computing all elements of \mathcal{M} (although it does not answer the question when the algorithm goes on indefinitely without producing an element of \mathcal{M} in a finite number of steps, cf. [3, Sect. 6]).

8. FURTHER PROPERTIES OF \mathcal{M}

In this section we examine in greater depth the construction of the set \mathcal{M} . We continue to assume throughout that $f, f': X \rightarrow Y$ satisfy conditions G1–G3 of 5.6.

8.1. DEFINITION. Given $e \in E(Df)$ let $\{p_i, i \leq n\}$ be a construction of approximate invariants all starting with edge e (7.5). Recall that each p_i is (f, f') -reduced with p_i an initial segment of $p_{i+1} = p_i \cdot [\tau p_i, \iota x_{i+1}]_{Df} \cdot x_{i+1}$. Here $f_*(p_i) = y_1 \cdots y_i$ (reduced), $f'_*(p_i) = y_1 \cdots y_{s_i}$, $f(x_i) = y_i$, $i \leq s_i$, and the inequality is strict for $i < n$. Also $p_n \in \mathcal{M}$ iff $s_n = n$. We call the construction $\{p_i, i \leq n\}$ *standard* if for each $1 \leq i \leq n-1$, x_{i+1} is chosen so that p_{i+1} is of minimal length (so p_1 is of minimal length; p_2 is of minimal length subject to choice of $x_1 \cdots$; p_{i+1} is of minimal length subject to choice of $x_1, x_2, \dots, x_i; \dots$).

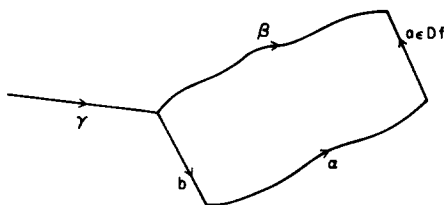
Let \mathcal{M}_e denote the set of (f, f') reduced minimal invariants p with initial edge e . If $e \in Df$ and $p \in \mathcal{M}_e$, we call p *standard* if $p = p_n$, where $\{p_i, i \leq n\}$ is a standard sequence of approximate invariants (7.6).

Similar definitions are made if $e \in Df'$.

8.2. LEMMA. Suppose $\{p_i, i \leq n\}$ is a standard construction of approximate invariants where the initial edge of all p_i is in Df . Let $a \in E(Df)$ be such that $\tau(p_n) = \iota(a)$. Then $p_n \cdot a$ is (f, f') -reduced provided $n > 1$ or if $n = 1$ and $p_1 \notin \mathcal{M}$.

Proof. $p_n = p_{n-1} \cdot [\tau p_{n-1}, \iota x_n]_{Df} \cdot x_n$ with $x_n \notin Df$. Since p_n is given f -reduced, it follows that $p_n \cdot a$ is f -reduced. If $x_n \notin Df'$, then $f'(x_n) \neq f'(\bar{a})$, since $f'|(X - E(Df'))$ is an immersion. Thus $p_n \cdot a$ is f' reduced if $x_n \notin Df'$.

We may assume then that $x_n \in E(Df')$ and $p_n \cdot a$ is not f' -reduced. We may then write $p_n = \gamma \cdot b \cdot \alpha$, where $\alpha = [\tau b, \tau p_n]_{Df'}$, $f'(b) = f'(\bar{a})$, and α is a non-trivial geodesic in Df' . Let $\beta = [\iota b, \tau \alpha]_{Df'}$.



We claim that $b \in E(Df)$. If not, then $b \notin E(Df) \cup E(Df')$. Thus $b = x_i$ for some $i < n$. We have $f'_*(\beta) = f'_*(b \cdot \alpha \cdot a) = 1$. By G3 we deduce that $f_*(\beta) = f_*(b \cdot \alpha) = f_*(b \cdot \alpha)$ as reduced words in $\pi_1(Y)$. Let x' be the first edge of β . Then $x' \in E(Df')$ and $f(x') = f(b)$ and $\iota(x') = \iota(b)$. But x' and b are both in $X - E(Df)$ and f is an immersion on $X - E(Df)$. It follows that $x' = b$, whence $b \in Df'$, a contradiction. Thus $b \in E(Df)$ as claimed.

As in the preceding paragraph, we deduce that $f_*(\beta) = f_*(b\alpha a) = f_*(\alpha)$ as reduced words in $\pi_1(Y)$. Also $f_*(\gamma\beta) = f_*(\gamma\beta\bar{a}) = f_*(\gamma b\alpha) = f_*(p_n)$.

There is a special case we must consider first, when γ is empty, so $b \in Df$ is the initial edge of all p_i . In this case, since $b \cdot \alpha$ is an approximate invariant and α is a geodesic in Df' , we deduce that $l(\alpha) = 1$, $\alpha = x_1 \in E(Df')$, $f'(b) = f(x_1)$, so $b \cdot \alpha \in \mathcal{M}$ and $n = 1$, $p_1 = b \cdot \alpha$.

We assume now that γ is not empty, so γ begins with the same edge in Df as all the p_i .

Let x be the first edge of α , $x \in E(Df')$, and let x' be the first edge of β , $x' \in E(Df')$. Then $\gamma \cdot b \cdot x = p_i$ for a unique index i , $i \leq n$. Let $p'_i = \gamma \cdot x'$. Since $f_*\beta = f_*\alpha$, we deduce that $f_*(p'_i) = f_*(p_i)$.

We claim that p'_i is (f, f') -reduced. If this were established, note that $l(p'_i) = l(p_i) - 1$, so we would get $\{p_1, \dots, p_{i-1}, p'_i\}$, a construction of approximate invariants (7.5) of shorter length than $\{p_1, \dots, p_i\}$, contradicting the assumption that $\{p_j, j \leq n\}$ was standard.

We show first that p'_i is reduced. If γ ends in an edge of $X - E(Df')$, this is clear since $x' \in E(Df')$. If γ ends in an edge $z \in E(Df')$, write $\gamma = \delta \cdot z$. If $z = \bar{x}'$, then $f_*(\delta) = f_*(p'_i) = f_*(p_i)$. But $f_*(p_i)$ has length i and $f_*(\delta)$ has length $< i$, so a contradiction is achieved. The same argument shows in fact that p'_i is f -reduced.

Since $x' \in E(Df')$ and γ is f' reduced and $\gamma \cdot x'$ is reduced, it follows that $p'_i = \gamma \cdot x'$ is f' -reduced. Hence p'_i is (f, f') -reduced. The proof of the claim is complete and 8.2 follows.

8.3. PROPOSITION. *Let $\{p_i, i \leq n\}$ be two constructions of approximate invariants starting with the same edge $e \in Df$. Assume $\{p_i, i \leq n\}$ is standard. Then $f_*(p_n) = f_*(p'_n)$ and $f'_*(p_n)$ is an initial segment of $f'_*(p'_n)$.*

Proof. It may happen that $n = 1$ and p_1 is invariant, so $p_1 = e \cdot x_1$, $x_1 \in E(Df')$, $f(x_1) = f'(e)$. In this case $p'_1 = e \cdot [\tau e, \iota x_1]_{Df} \cdot x'_1$ is (f, f') reduced with $f'(e) = f(x'_1)$. Thus $f_*(p'_1) = f(x'_1) = f'(e) = f(x_1) = f_*(p_1)$ and $f'_*(p'_1)$ has initial edge $f'(e) = f_*(p_1)$. Thus the proposition is true in the degenerate case excluded by 8.2. We shall assume now that $n > 1$ or that $n = 1$ and p_1 is not invariant, so 8.2 is applicable and proceed by induction on n .

If $\alpha = [\tau p_n, \tau p'_n]_{Df}$, then it follows from 8.2 and G1, G2 that $p_n \cdot \alpha$ is (f, f') -reduced. Consider first the case $n = 1$. Here $f_*(p_1 \cdot \alpha) = f_*p_1 = f'(e) = f_*(p'_1)$, and $\tau(p_1 \cdot \alpha) = \tau(p'_1)$. We get from G3 that $f'_*(p_1 \cdot \alpha) = f'_*(p'_1)$. Thus $f'_*(p_1)$ is an initial segment of $f'_*(p'_1)$ and the induction starts.

Assume that $f_*(p_{n-1}) = f_*(p'_{n-1})$ and $f'_*(p_{n-1})$ is an initial segment of $f'_*(p'_{n-1})$. We shall show the corresponding conclusions with $n - 1$ replaced by n . Since neither p_{n-1} nor p'_{n-1} is invariant (otherwise the construction 7.5 terminates at $n - 1$) and $f'_*(p_{n-1})$ is an initial segment of

$f'_*(p'_{n-1})$ we get $y_n = y'_n$, where $f_*(p_n) = y_1 \cdots y_n$, $f_*p'_n = y'_1 \cdots y'_n$. Thus $f_*(p_n) = f'_*(p'_n)$. It remains to show that $f'_*(p_n)$ is an initial segment of $f'_*(p'_n)$.

If $\alpha = [\tau p_n, \tau p'_n]_{Df}$, then it follows from 8.2 and G1, G2 that $p_n \cdot \alpha$ is (f, f') -reduced. Since $f_*(p_n \cdot \alpha) = f_*(p_n) = f'_*(p'_n)$ and $\tau(p_n \cdot \alpha) = \tau(p'_n)$, we get from G3 that $f'_*(p_n \cdot \alpha) = f'_*(p'_n)$. Thus $f'_*(p_n)$ is indeed an initial segment of $f'_*(p'_n)$ and the proof is complete.

8.4. COROLLARY. *Let $\{p_i, i \leq n\}$ and $\{p'_i, i \leq n\}$ be two standard constructions starting with the same edge $e \in E(Df)$. Then $f_*(p_n) = f'_*(p'_n)$, $f'_*(p_n) = f'_*(p'_n)$ and $\tau(p_n) = \tau(p'_n)$.*

Proof. Since $f'_*(p_n)$ and $f'_*(p'_n)$ are initial segments of each other by 8.3, it follows that they are equal. It follows that, if $\alpha = [\tau(p_n), \tau(p'_n)]_{Df}$, $f'_*(\alpha)$ is trivial. Since $f' \mid Df$ is an immersion, α is trivial, and $\tau(p_n) = \tau(p'_n)$.

8.5. DEFINITION. If p and p' are two paths in X , we write $p \doteq p'$ if $\iota(p) = \iota(p')$, $\tau(p) = \tau(p')$, $f_*(p) = f'_*(p')$ and (of necessity, from G3) $f'_*(p) = f'_*(p')$. We may restate the conclusion of 8.4 in this notation as $p_n \doteq p'_n$.

8.6. THEOREM. *If $p, p' \in \mathcal{M}_e$ with $e \in E(Df)$ and if p and p' are standard, then $p \doteq p'$.*

Proof. $p = p_n$ and $p' = p'_m$, where $\{p_i, i \leq n\}$ and $\{p'_i, i \leq m\}$ are standard constructions (7.6). Suppose $n \leq m$. Then by 8.3 $f_*(p_n) = f'_*(p'_n)$ and $f'_*(p_n)$ is an initial segment of $f'_*(p'_n)$ and $f'_*(p'_n)$ is an initial segment of $f'_*(p_n)$. Thus $f'_*(p'_n) = f'_*(p_n) = f'_*(p_n) = f'_*(p'_n)$ and it follows that $p'_n \in \mathcal{M}$, so $n = m$. From 8.4 it follows that $p \doteq p'$.

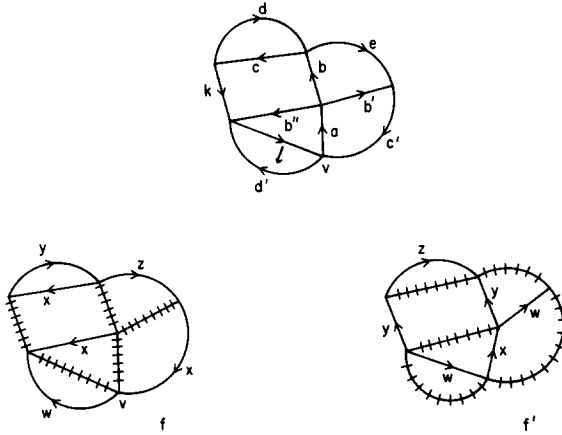
8.7. THEOREM. *If $p, p' \in \mathcal{M}_e$ with $e \in E(Df)$ and if p is standard, then $f_*(p)$ is an initial segment of $f'_*(p')$.*

Proof. Let $p = p_n$ and $p' = p'_m$ where $\{p_i, i \leq n\}$ and $\{p'_i, i \leq m\}$ are approximate invariants by 7.6. Suppose first that $m \leq n$. It follows from 8.3 that $f_*(p_m) = f'_*(p'_m)$ and $f'_*(p_m)$ is an initial segment of $f'_*(p'_m) = f'_*(p'_m)$. But by a length computation this implies p_m is invariant, so $m = n$.

Thus we may assume $n \leq m$.

By 8.3, $f_*(p_n) = f'_*(p'_n)$. Since the latter is an initial segment of $f'_*(p')$, the result follows.

8.8. EXAMPLE. Consider the graphs drawn below.



One checks that G1–G3 are satisfied. At vertex v , φ is given by $x \rightarrow xw$, $y \rightarrow w^{-1}yzy^{-1}x^{-1}$, $z \rightarrow xyw^{-1}x^{-1}$, $w \rightarrow w$. One has here $\mathcal{M}_a = \{abcde, ab'c'd', ab''\}$ and $f_*(abcde) = xyz$, $f_*(ab'c'd') = xw$, $f_*(ab'') = x$. Here ab'' is standard and we see directly that the conclusion of 8.7 is valid.

8.9. EXAMPLE. Review Example 6.8. The standard invariants are ad, bc, da, cb and their “bars.”

9. FINITENESS THEOREM

We shall extract from \mathcal{M} a finite subset which suffices to generate all fixed points of φ . As always, $f, f': X \rightarrow Y$ continue to satisfy G1–G3 of 5.6.

9.1. DEFINITION. Let \mathcal{M}' be the following subset of \mathcal{M} :

- (1) If $fe = f'e$, $e \in \mathcal{M}'$.
- (2) If $e \in E(Df)$ or $e \in E(Df')$ and if $\mathcal{M}_e \neq \emptyset$, choose one standard path $p_e \in \mathcal{M}_e$ (8.1) and for this choice, set $p_e \in \mathcal{M}'$. Observe that if p'_e is another standard path in \mathcal{M}_e , then by 8.6, $p_e \cong p'_e$.

We shall prove our finiteness results using the choice of \mathcal{M}' just made.

9.2. Remark. If $p \in \mathcal{M}$ is standard, we do not know if \bar{p} is also standard.

9.3. LEMMA. Let p, p' be invariant paths in X with $\iota(p) = \iota(p')$ and $f_*(p) = f_*(p')$. Then $\tau(p) = \tau(p')$, so $p \cong p'$.

Proof. Let $\alpha = [\tau p, \tau p']_{Df}$. Then $f_*(p \cdot \alpha) = f_*(p')$. Since $p \cdot \alpha$ and p' have the same initial and terminal vertices, it follows from G3 that $f'_*(p\alpha) = f'_*(p')$. Since p and p' are invariant, $f'_*(p) = f'_*(p')$. Thus $f'_*(\alpha)$ is trivial. However, $f' \mid Df$ is an immersion (G1 and G2), so α must be the trivial path (1.6a). Thus $\tau(p) = \tau(p')$.

9.4. THEOREM. $f_*(\langle \mathcal{M}' \rangle \cap \pi_1(X, v)) = f_*(\langle \mathcal{M} \rangle \cap \pi_1(X, v))$.

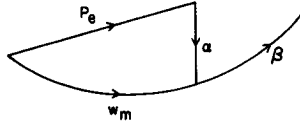
Proof. Given $w \in \mathcal{M}$, we shall show by induction on $l(f_*(w))$ that there is a path $p_1 p_2 \cdots p_n = p$ in X , $p_i \in \mathcal{M}'$, with $p \cong w$.

If $w \in E(X)$ and $fw = f'w$, then $w \in \mathcal{M}'$. If $l(f_*(w)) = 1$ and the initial edge e of w is in Df , then of necessity $w = e \cdot x_1$, where $f'(e) = f(x_1)$ (7.5) and $x_1 \in E(Df')$ (7.7). We claim $w = p_e$. For if $e \cdot x'_1 \in \mathcal{M}_e$, then $f(x'_1) = f(x_1) = f'(e)$. But since $f \mid (X - E(Df))$ is an immersion, $x_1 = x'_1$. Thus $w = p_e \in \mathcal{M}'$.

The case $l(f_*(w)) = 1$ with initial edge of w in Df' is handled similarly. Thus the induction starts.

Assume the inductive assertion for all $w' \in \mathcal{M}$ with $l(f_*(w')) \leq n-1$, and let $w \in \mathcal{M}$ with $l(f_*(w)) = n > 1$. It follows that the first edge e of w is in $Df \cup Df'$. Without loss of generality, assume $e \in E(Df)$. Let $l(f_*(p_e)) = m$, so $m \leq n$ (8.7). In addition $f_*(p_e)$ is an initial segment of $f_*(w)$ (8.7). If $m = n$, then $f_*(p_e) = f_*(w)$, so by 9.3, $p_e \cong w$ and we are done.

If $m < n$ let $\alpha = [\tau(p_e), \tau(w_m)]_{Df}$, where $\{w_i, i \leq n\}$ is a sequence of approximate invariants for $w = w_n$ (7.6). Since



$f_*(p_e) = f_*(w_m)$ (8.3), we have $f_*(w_m \cdot \bar{\alpha}) = f_*(p_e)$. Then G3 implies $f'_*(w_m \cdot \bar{\alpha}) = f'_*(p_e)$. From this it follows that $w_m \cdot \bar{\alpha}$ is invariant and hence so is $\alpha \cdot \beta$, where β is the tail segment of w after w_m , $w = w_m \cdot \beta$. Observe that $l(f_*(w_m \cdot \bar{\alpha})) < n$ and $l(f_*(\alpha\beta)) < n$. By 6.4 there are (f, f') reduced paths γ_1 and γ_2 in X with $\gamma_1 \cong w_m \cdot \bar{\alpha}$ and $\gamma_2 \cong \alpha \cdot \beta$. We may write γ_1, γ_2 as path products of certain paths p in \mathcal{M} . Of necessity those p occurring satisfy $l(f_*(p)) \leq \max_{i=1,2} l(f_*(\gamma_i)) < n$. Thus the induction hypothesis implies there are paths δ_1, δ_2 , path products of paths in \mathcal{M}' , with $\gamma_i \cong \delta_i$. Thus $\delta_1 \cdot \delta_2 \cong \gamma_1 \cdot \gamma_2 \cong (w_m \cdot \bar{\alpha}) \cdot (\alpha \cdot \beta) \cong w_m \cdot \beta = w$, and w is a product of paths in \mathcal{M}' . The induction is complete.

To prove the theorem, assume $w \in \pi_1(X, v)$ and w is a path product of paths in \mathcal{M} . It follows that there is a product p of paths in \mathcal{M}' with $p \cong w$. Thus $f_*(w) \in f_*(\langle \mathcal{M}' \rangle \cap \pi_1(X, v))$, and the proof of 9.4 is complete.

9.5. COROLLARY. $\text{Fix}(\varphi)$ is a finitely generated free group.

Proof. By 6.1.1 and 6.7, $\text{Fix}(\varphi) = f_*(\langle \mathcal{M} \rangle \cap \pi_1(X, v))$. By 8.9 the latter group is $f_*(\langle \mathcal{M}' \rangle \cap \pi_1(X, v))$. But \mathcal{M}' is a finite set, so $\langle \mathcal{M}' \rangle$ and $\pi_1(X, v)$ are finitely generated subgroups of $\pi_1(Z)$. By Howson's theorem [10, 5.6], $\langle \mathcal{M}' \rangle \cap \pi_1(X, v)$ is finitely generated. Thus $\text{Fix}(\varphi)$ is finitely generated.

9.6. **EXAMPLE.** Refer again to 5.7, 6.8, 8.9. $\mathcal{M}' = \{e, \bar{e}, ad, d\bar{a}, cb, b\bar{c}, bc, \bar{c}\bar{b}, da, \bar{a}\bar{d}\}$. Then $\langle \mathcal{M}' \rangle \cap \pi_1(X, v) = \langle ad, ec\bar{b}\bar{e} \rangle$ from the algorithmic form of Howson's theorem given by Stallings [10, 5.6]. It follows that $\text{Fix}(\varphi) = \langle x, yxy^{-1} \rangle$ as asserted in 6.8.

9.7. **PROPOSITION.** If $\text{Fix}(\varphi) \neq \{1\}$, then $\text{rank}(\text{Fix}(\varphi)) - 1 \leq 2(\#E(X) - 1) \cdot (h_1(X) - 1)$, where $h_1(X)$ is the first betti number of X .

Proof. This is a consequence of 9.4 and the H. Neumann inequality (see [2, 4.1]), which in our context says that if $\langle \mathcal{M}' \rangle \cap \pi_1(X, v) \neq \{1\}$ then $\text{rank}(\langle \mathcal{M}' \rangle \cap \pi_1(X, v)) - 1 \leq 2(\text{rank}(\langle \mathcal{M}' \rangle) - 1) \cdot (h_1(X) - 1)$. Observing that $\#(\mathcal{M}') \leq \#E(X)$, the assertion follows.

9.8. **THEOREM.** If G is a finitely generated group of automorphisms of the finitely generated free group F , then $F^G =: \{x \in F \mid gx = x \text{ for all } g \in G\}$ is a finitely generated subgroup of F .

Proof. Let $\{\phi_1, \phi_2, \dots, \phi_n\}$ be a finite set of generators for G . Then $F^G = \bigcap_{i=1}^n \text{Fix}(\phi_i)$. By 9.5 each $\text{Fix}(\phi_i)$ is finitely generated. From Howson's theorem it follows that F^G is finitely generated, and the proof is complete.

9.9. **Remark.** This result 9.8 settles the question raised in [1, p. 195], whether F^G is finitely generated, for finitely generated free groups F , in the case G is a finitely generated group of automorphisms of F . The general case, whether F^G is finitely generated for arbitrary G and finitely generated free F , is still open.

9.10. **Remark.** If we could prove that \bar{p} is standard if p is standard, then the techniques of [3, Sect. 8] would show $\text{rank } \text{Fix}(\varphi) \leq h_1(X)$. In fact in all the examples we have computed as well as in the references [1, 6, 3], the stronger inequality $\text{rank } \text{Fix}(\varphi) \leq h_1(Y)$ holds. We would like to inquire whether this stronger inequality holds in general.

9.11. **Remark.** In an earlier version of this article we asked whether the following generalization of Corollary 9.5 was true: if $f, g: F \rightarrow F_1$ are two homomorphisms of finitely generated free groups and if $\mathcal{E} = \{x \in F \mid \alpha(x) = \beta(x)\}$ then was $f(\mathcal{E})$ finitely generated? That this is not the case in general may be seen by letting F be free with basis $\{x, y, z\}$ and F_1 be free with basis $\{a, b\}$ and defining $f(x) = 1$, $f(y) = a$, $f(z) = b$, $g(x) = a$, $g(y) = 1$, and $g(z) = b^{-1}$. If one sets $u_n = z^n x z^{-2n} y z^n$, $n \in \mathbb{Z}$, then $f(u_n) = b^{-n} a b^n = g(u_n)$ so $u_n \in \mathcal{E}$. It is not difficult to verify from this that

$f(\mathcal{E}) = \langle b^{-n}ab^n, n \in \mathbb{Z} \rangle$, which is not finitely generated. This example is due to John Stallings, based on a suggestion of Craig Squier's.

9.12. THEOREM. *If G is a finitely generated discrete subgroup of $PSL_2(\mathbb{R})$ and $f: G \rightarrow G$ is an automorphism of G , then $\text{Fix}(f)$ is finitely generated.*

Proof. By a result of Selberg's [8] G contains a subgroup N of finite index without elliptic elements, whence N is torsion free. By passage to a subgroup of finite index, we may assume in addition that N is a characteristic subgroup of G . Thus $f|N = f_1$ is an automorphism of N . Since N acts properly discontinuously and freely on the upper half plane H , it follows that the orbit space $N \backslash H$ is a two dimensional manifold with finitely generated fundamental group N . There are two cases to consider.

Case 1. $N \backslash H$ is not closed. In this case N is free, so by Corollary 9.5, $\text{Fix}(f_1)$ is finitely generated. Since $\text{Fix}(f)$ is a finite extension of $\text{Fix}(f_1)$, it follows that $\text{Fix}(f)$ is finitely generated.

Case 2. $N \backslash H$ is closed. Since $N \backslash H$ is a space of type $K(N, 1)$ (assuming $N \neq \{1\}$) there is a homotopy equivalence g of $N \backslash H$ which may be assumed to preserve a base point p , such that the map induced by g on the fundamental group is f_1 . Furthermore g is homotopic to a homeomorphism h of $N \backslash H$ (see [12, p. 61, Lemma 1.4.3]) through a homotopy which preserves the base point p , so $f_1 = h_*$ as automorphisms of $\pi_1(N \backslash H, p)$. Now the result due to Jaco and Shalen [6] applies to show that $\text{Fix}(h_*)$ is finitely generated. It follows as in Case 1 that $\text{Fix}(f)$ is finitely generated. This completes the proof.

9.13. Remark. It follows from Theorem 9.12) that the fixed points of an automorphism of a finitely generated Fuchsian or NEC group [7, p. 146] are finitely generated. In effect, this has been conjectured in 1974 by Jaco [5, p. 307].

APPENDIX

We shall prove the form of M. Hall's theorem that was needed in the proof of 3.6.

THEOREM. *If $f: X \rightarrow Y$ is an immersion of finite graphs, then there is a commutative diagram*

$$\begin{array}{ccc} & X \subset Z & \\ f \swarrow & & \searrow g \\ & Y & \end{array}$$

where X is a subgraph of Z and g is a finite covering. If X and Y are connected, Z may be chosen to be connected.

Proof. Let $Y \xrightarrow{h} Y_1$ collapse all the vertices of Y to one vertex. Clearly h is an immersion. Thus $h \circ f: X \rightarrow Y_1$ is an immersion of X into a graph Y_1 with a single vertex. We apply Stallings's version of M. Hall's theorem [10, 6.1] to get a commutative diagram

$$\begin{array}{ccc} X & \subset & Z_1 \\ & \searrow & \swarrow g_1 \\ & Y_1 & \end{array}$$

where X is a subgraph of Z_1 and g_1 is a finite covering (and in fact $V(X) = V(Z_1)$). Now form the fibre product Z of h and g_1 and use the universal property of fibre products to get a commutative diagram

$$\begin{array}{ccccc} & & X & & \\ & \searrow & & \searrow & \\ & & Z & \longrightarrow & Z_1 \\ & \searrow f & \downarrow g & & \downarrow g_1 \\ & & Y & \xrightarrow{h} & Y_1 \end{array}$$

Since $X \subset Z_1$ it follows that $X \rightarrow Z$ is injective. Since g_1 is a covering it follows that g is a covering. The last statement in the theorem is trivial, so this completes the proof.

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